Proof of Euler's Formula : $e^{iy} = \cos y + i \sin y$

Power series for exponential: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\therefore e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{(x+iy)^k}{k!}$

Now let $z = 0 + iy$.

$e^{iy} = e^{0+iy} = \sum_{k=0}^{\infty} \frac{(0+iy)^k}{k!} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k y^k}{k!}$

We split up the odd + even terms. For even k, $i^k = -1$ or 1, + for even k, its i or $-i$.

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$$e^{iy} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right)$$

Converges to cos y

Converges to sin y

$$\therefore e^{iy} = \cos(y) + i \sin(y)$$ QED.

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